# The Quantum Field Theory of Quadratic Gravity

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My work is with Gabriel Menezes:
arXiv:1712.04468 , arXiv:1804.04980, arXiv:1812.03603
**arXiv:1908.02416, arXiv:1908.04170**, **aXiv:2003.09047**…
But this understanding builds on the past work of many others

Amherst Center for Fundamental Interactions
*Physics at the interface: Energy, Intensity, and Cosmic frontiers*
University of Massachusetts Amherst

# Connections:

Early pioneers: **Stelle**, Fradkin-Tsetlyn, Adler, Zee, Smilga,
Tomboulis, Hasslacher-Mottola,
**Lee-Wick,** Coleman, Boulware-Gross….

Present activity: Einhorn-Jones, Salvio-Strumia, Holdom-Ren,
Donoghue-Menezes, Mannheim, Anselmi
Odintsov-Shapiro, Narain-Anishetty…

In the neighborhood: Lu-Perkins-Pope-Stelle, ‘t Hooft,
Grinstein-O’Connell-Wise ….

# Outline

1) **Overview of key features**
   - Spectrum
   - Unitarity
   - Causality
   - Stability

2) **Causality in more detail**

3) **Unitarity in more detail**

4) **Summary/ open issues**

# <u>**Quadratic gravity**</u>:

$$S_{\rm quad} = \int d^4x \sqrt{-g} \left[ \frac{2}{\kappa^2} R + \frac{1}{6 f_0^2} R^2 - \frac{1}{2\xi^2} C_{\mu\nu\alpha\beta} C^{\mu\nu\alpha\beta} \right]$$

Renormalizeable QFT for quantum gravity
- the most conservative version of quantum gravity

<u>BUT</u>:

$$R \sim \partial^2 g \qquad\qquad R^2 \sim \partial^2 g \partial^2 g$$

Higher derivative theories have “issues” and mythology

**Bottom line**: Find unitary theory, appears stable near Minkowski
- but with Planck scale causality violation/uncertainty

# Caution: not all approaches need be equivalent

**Usual way we teach/discuss theories:**

1) Classical physics and solutions
2) Canonical Hamiltonian quantization of free field theory
3) Add interactions
4) Repeat with Lagrangian Path Integrals

**Here – reverse pathway:**

1) Start with Lorentzian Lagrangian Path Integral
2) Include interactions with matter (also leading self interactions)
3) Then, analyze gravitational sector
4) Limits to standard EFT at low energy (and classical physics)

Reverse pathway is like the approach to Electroweak theory

Our understanding of the equivalence is based on standard theories

# Why this path?

Spectrum becomes clear at first step
- only stable state is massless graviton

No need for canonical quantization of unstable ghost

Only stable states appear in unitarity sum
- no ghosts, only their decay products

Low energy limit is usual gravitational EFT
- with usual stability properties

# Also Lorentzian vs Euclidean

**This equivalence is not a sacred principle**

- changes in causality/spectrum
- especially problematic for gravity !

**This path starts with Lorentzian PI**

Note: Anselmi takes different path – starting from Euclidean

- perhaps the paths will merge someday

# Spectrum

$$S_{\rm quad} = \int d^4x\sqrt{-g}\left[\frac{2}{\kappa^2}R + \frac{1}{6f_0^2}R^2 - \frac{1}{2\xi^2}C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta}\right]$$

- The graviton propagator gets modified by $q^4$ terms, roughly

$$iD(q) = \frac{i}{q^2 - q^4/M^2} = \frac{i}{q^2} - \frac{i}{q^2 - M^2}$$

- Spin zero portion leads to either a tachyon or a normal resonance
  - depends on sign of $f_0^2$
- Spin two portion leads to either a tachyon or an **unstable ghost**
  - depends on sign of $\xi^2$

- Choose signs to **avoid tachyons** (i.e no poles at spacelike momenta)

**Need to focus on spin-two propagator and find spectrum**

# The spin-two propagator (including self-energy)

$$D_{\mu\nu\alpha\beta}(q^2) = \mathcal{P}^{(2)}_{\mu\nu\alpha\beta} D(q^2)$$

$$D^{-1}(q^2) = q^2 + i\epsilon - \frac{\kappa^2 q^4}{2\xi^2(\mu)} - \frac{N_{\text{eff}}}{640\pi^2}\kappa^2 q^4 \ln\left(\frac{-q^2 - i\epsilon}{\mu^2}\right)$$

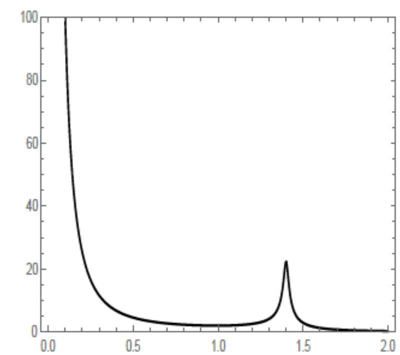
General structure:

$$iD(q) = \frac{i}{q^2 + i\epsilon - \frac{q^4}{M^2} + \Sigma(q)} \quad . \qquad \text{with } \mathrm{Im}\ \Sigma(q) = \gamma(q) \qquad \gamma(q) \geq 0$$

Massless pole is usual graviton

The high mass pole carries **two** minus sign differences:

$$\begin{aligned} iD_F(q) &= \frac{i}{q^2 - \frac{q^4}{M^2} + i\gamma(q)} \\ &= \frac{i}{\frac{q^2}{M^2}[M^2 - q^2 + i\gamma(q)(M^2/q^2)]} \\ &\sim \frac{-i}{q^2 - M^2 - i\gamma_M} \quad . \end{aligned}$$



**Interpretation:**

This is different from normal resonance

$$iD_F \sim_{q^2 \sim m^2} = \frac{i}{q^2 - m^2 + i\gamma_m} \sim \frac{i}{q^2 - \left(m_r - i\frac{1}{2}\Gamma\right)^2}$$

Here we have

$$\sim \frac{-i}{q^2 - M^2 - i\gamma_M}$$

This is **time-reversed** version of a resonance propagator

- time reversal is anti-unitary

Still corresponds to decaying particle

Important for unitarity – **imaginary parts are the same**

$$iD(q) \sim \frac{Zi}{q^2 - m^2 + iZ\gamma} \qquad \mathrm{Im}[D(q)] \sim \frac{-\gamma}{(q^2 - m^2)^2 + \gamma^2}$$

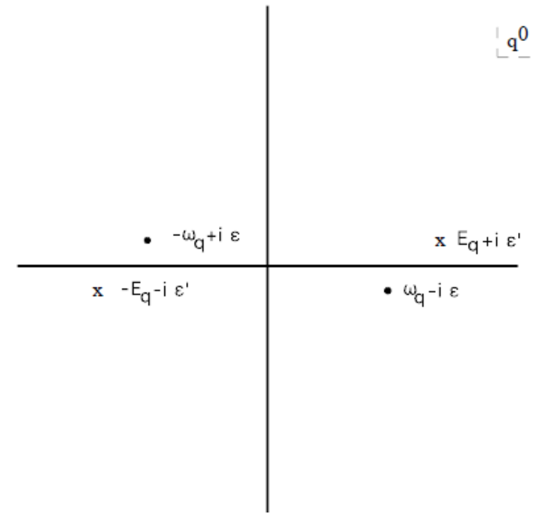
# Propagator – time orderings

$$iD_F(x) = D_F^{\rm for}(x)\theta(t) + D_F^{\rm back}(x)\theta(-t)$$

Note **energy flow**, and also decay lifetime

$$D^{\rm for}(t,\vec{x}) = \int \frac{d^3q}{(2\pi)^3}\left[\frac{e^{-i(\omega_q t - \vec{q}\cdot\vec{x})}}{2\omega_q} - \frac{e^{i(E_q t - \vec{q}\cdot\vec{x})}}{2(E_q + i\frac{\gamma}{2E_q})}e^{-\frac{\gamma t}{2E_q}}\right]$$

$$D^{\rm back}(t,\vec{x}) = \int \frac{d^3q}{(2\pi)^3}\left[\frac{e^{i(\omega_q t - \vec{q}\cdot\vec{x})}}{2\omega_q} - \frac{e^{-i(E_q t - \vec{q}\cdot\vec{x})}}{2(E_q + i\frac{\gamma}{2E_q})}e^{-\frac{\gamma|t|}{2E_q}}\right]$$



## **Merlin modes:**

-Merlin (the wizard in the tales of King Arthur) ages backwards

*"Now ordinary people are born forwards in Time, if you understand what I mean, and nearly everything in the world goes forward too. (...) But I unfortunately was born at the wrong end of time, and I have to live backwards from in front, while surrounded by a lot of people living forwards from behind."*

T. H. White *Once and Future King*

Note, there is a key distinction with usual nomenclature "ghosts"

- ghost is anything with a minus sign in the numerator
- these Merlin modes refer to crucial sign $-i\gamma$ in denominator in addition

# Unitarity of unstable particles:

$$\langle f|T|i\rangle - \langle f|T^\dagger|i\rangle = i\sum_j \langle f|T^\dagger|j\rangle\langle j|T|i\rangle$$

**Who counts in unitarity relation?**

- Veltman 1963
- **only stable particles count**
- they form asymptotic Hilbert space
- **do not** make any cuts on unstable resonances

UNITARITY AND CAUSALITY IN A RENORMALIZABLE FIELD THEORY WITH UNSTABLE PARTICLES

M. VELTMAN *)

This looks funny from free-field quantization

- interaction removes states from the Hilbert space

Also, we know some states are almost stable

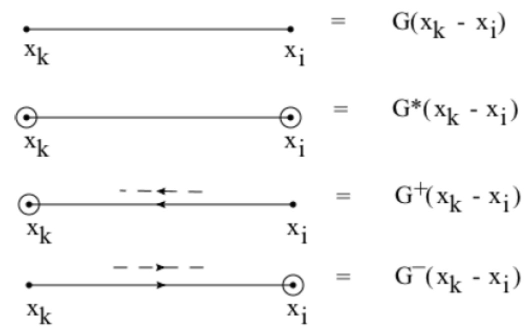
- can treat them as essentially stable
- Narrow Width Approximation (NWA)

But of course, Veltman is correct

# <u>Formal proof of unitarity with unstable ghosts</u>

**With G. Menezes arXiv:1908.02416 in PRD**

**Follows Veltman**:
- circling rules
- largest time equation
- turns into derivation of cutting rules




Only difference is **energy flow**

$$\begin{aligned} -iG^*(x-x') &= \Theta(x_0-x_0')G^-(x-x') + \Theta(-x_0+x_0')G^+(x-x') \\ -i\widetilde{G}^*(x-x') &= \Theta(x_0-x_0')\widetilde{G}^-(x-x') + \Theta(-x_0+x_0')\widetilde{G}^+(x-x') \\ -i\widetilde{G}^*_{\rm GH}(x-x') &= \Theta(x_0-x_0')\widetilde{G}^+_{\rm GH}(x-x') + \Theta(-x_0+x_0')\widetilde{G}^-_{\rm GH}(x-x') \end{aligned}$$

Important point - all steps in Minkoswki space
- no analytic continuation employed

**Formalizes early work by Lee-Wick**

# Causality

Known since Lee-Wick and Coleman that such propagators lead to micro-causality violation

Traced to backwards-in-time propagation of Merlin
- dueling **arrows of causality**

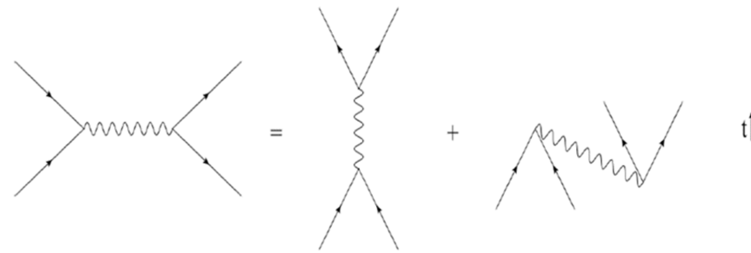
But limited to time scales proportional to lifetime

For gravity this is inverse Planck scale

Lee, Wick
Coleman
Grinstein, O'Connell, Wise
Alvarez, Da Roid, Schat, Szynkman

# Phenomenology

**Vertex displacements**: (ADSS)
- look for final state emergence
- before beam collision

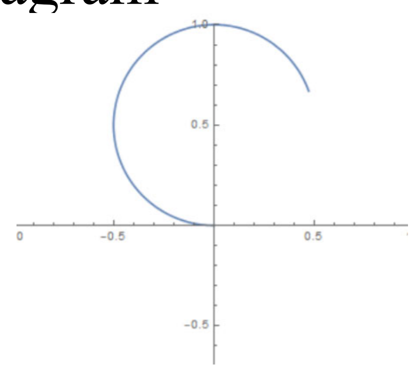



**Form wavepackets – early arrival** (LW, GOW)
- wavepacket description of scattering process
- some components arrive at detector early

**Resonance Wigner time delay reversal**
- normal resonaces counterclockwise on Argand diagram

$$\Delta t \sim \frac{\partial \delta}{\partial E} \sim> 0$$

- Merlin modes are clockwise resonance



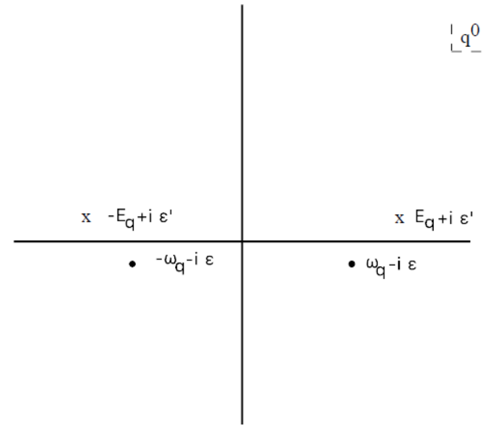

**For gravity, all are Planck scale**
- no conflict with experiment

# Stability:

See also Salvio;
Reis, Chapiro, Shapiro

Consider propagator with retarded BC:

$$\log\left(-\left[(q_0+i\epsilon)^2-\vec{q}^2\right]\right)=\log\left(-q^2-i\epsilon q_0\right)=\log|q^2|-i\pi\theta(q^2)\left(\theta(q_0)-\theta(-q_0)\right)$$




Again propagation in both directions:

$$D_{\rm ret}(t>0,\vec{x})=D^{(0)}_{\rm ret}(t>0,\vec{x})$$

$$D_{\rm ret}(t<0,\vec{x})\equiv D^{<}_{\rm ret}(t,\vec{x})=i\int\frac{d^3q}{(2\pi)^3}\left[\frac{e^{-i(E_qt-\vec{q}\cdot\vec{x})}}{2(E_q+i\frac{\gamma}{2E_q})}e^{-\frac{\gamma|t|}{2E_q}}-\frac{e^{i(E_qt-\vec{q}\cdot\vec{x})}}{2(E_q-i\frac{\gamma}{2E_q})}e^{-\frac{\gamma|t|}{2E_q}}\right]$$

Backwards perturbations have finite lifetime:

$$h_{\mu\nu}(t,x)=\int d^3x'\left[\int_{-\infty}^{t}dt'D^{(0)}_{\rm ret}(t-t',x-x')+\int_t^{\infty}dt'D^{<}_{\rm ret}(t-t',x-x')\right]J_{\mu\nu}(t',x')$$

No growing modes – no sign of Ostrogradsky instability
-logically this is likely related to unitarity

## **End of initial overview:**

Identifying spectrum early is key step

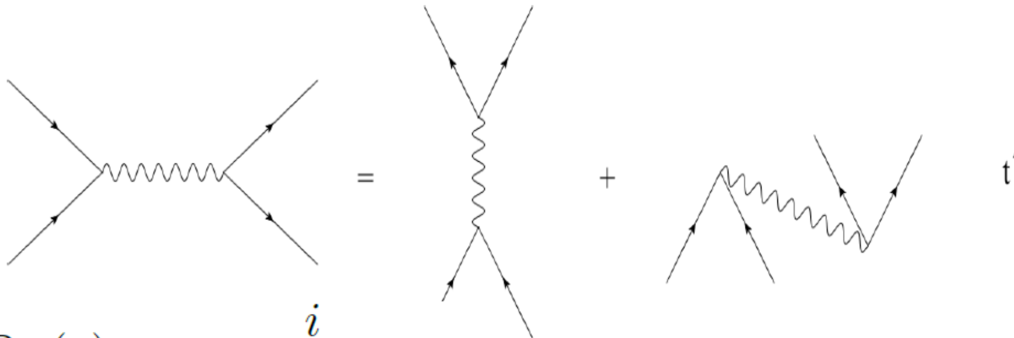
- decay removes ghost from asymptotic spectrum

Factors of $i$ are crucial

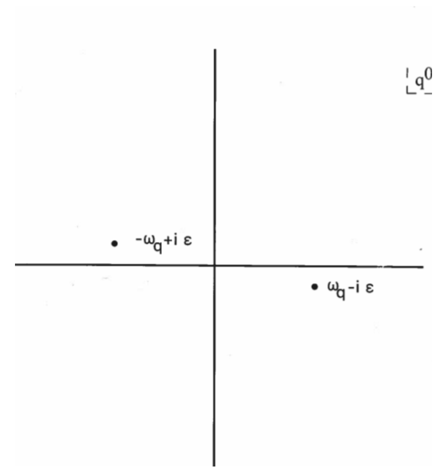
- two changes of $i$ in propagator

## More on causality

Causality is not really "cause before effect"




$$iD_F(q) = \frac{i}{q^2 - m^2 + i\epsilon}$$




Decompose into time orderings:

$$iD_F(x) = D_F^{\rm for}(x)\theta(t) + D_F^{\rm back}(x)\theta(-t)$$

Positive energies propagate forward in time
- backwards propagation is "negative energy"

But backward-in-time propagation shielded by uncertainty principle
$\Delta t \sim 1/\Delta E$

# Operators commute for spacelike separation

$$[\mathcal{O}(x), \mathcal{O}(x')] = 0 \quad \text{for} \quad (x - x')^2 < 0$$

Note: metric is (+,-,-,-)

PHYSICAL REVIEW VOLUME 95, NUMBER 6 SEPTEMBER 15, 1954

## Use of Causality Conditions in Quantum Theory

M. Gell-Mann, *Institute of Nuclear Studies and Department of Physics, University of Chicago, Chicago, Illinois*
M. L. Goldberger,* *Princeton University, Princeton, New Jersey*

AND

W. E. Thirring,† *Institute for Advanced Study, Princeton, New Jersey*

(Received May 24, 1954)

The limitations on scattering amplitudes imposed by causality requirements are deduced from the demand that the commutator of field operators vanish if the operators are taken at points with space-like separations. The problems of the scattering of spin-zero particles by a force center and the scattering of photons by a quantized matter field are discussed. The causality requirements lead in a natural way to the well-known dispersion relation of Kramers and Kronig. A new sum rule for the nuclear photoeffect is derived and the scattering of photons by nucleons is discussed.

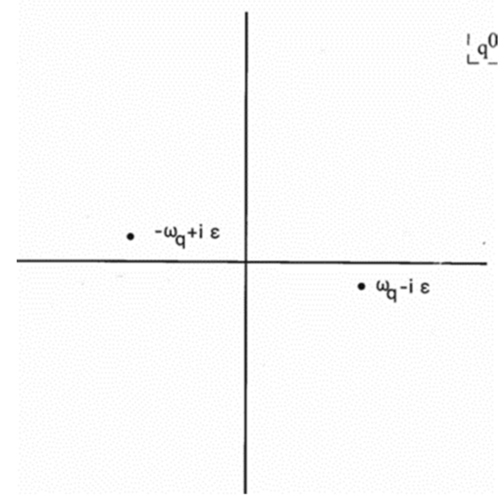
This requires negative energy part of propagator to accomplish

# But also – Arrow of Causality

What determines past lightcone and future lightcone?
- and why do all particles share this?

This comes from the $i\epsilon$

$$iD_F(q) = \frac{i}{q^2 - m^2 + i\epsilon}$$




Determines that positive energy propagates forward in time

## What if we used $e^{-iS}$ instead of $e^{iS}$?

Consider generating functions:

$$
\begin{aligned}
Z_\pm[J] &= \int [d\phi] e^{\pm i S(\phi, J)} \\
&= \int [d\phi] e^{\pm i \int d^4x [\frac{1}{2}(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2) + J\phi]} .
\end{aligned}
$$

Need to make this better defined – add

$$
e^{-\epsilon \int d^4x \phi^2/2}
$$

Solved by completing the square:

$$
Z_\pm[J] = Z[0] \exp\left\{ -\frac{1}{2} \int d^4x d^4y J(x)\ iD_{\pm F}(x-y) J(y) \right\}
$$

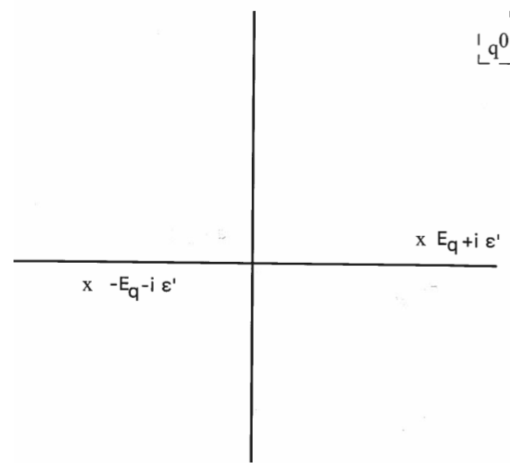
Yield propagator with specific analyticity structure

$$
iD_{\pm F}(x-y) = \int \frac{d^4q}{(2\pi)^4} e^{-iq\cdot(x-y)} \frac{\pm i}{q^2 - m^2 \pm i\epsilon}
$$

# Result is time-reversed propagator

$$iD_{-F}(x) = D_{-F}^{\text{for}}(x)\theta(t) + D_{-F}^{\text{back}}(x)\theta(-t)$$

"Positive energy" propagates backwards in time




$$D_{-F}^{\text{back}}(x) = \int \frac{d^3q}{(2\pi)^3 2E_q} e^{-i(E_q t - \vec{q}\cdot\vec{x})}$$

Use of this generating functional yields time reversed scattering processes

**Opposite arrow of causality**

# Time reversal is anti-unitary

Lagrangian can be invariant, but PI is not

$$Z_+[J] \to Z_-[J]$$

Note: Also can be found in canonical quantization
Changes

$$[\phi(t,x), \pi(t,x')] = i\hbar\delta^3(x-x')$$

to

$$[\phi(t,x), \bar{\pi}(t,x')] = -i\hbar\delta^3(x-x') \quad \text{with} \quad \bar{\pi} = \frac{\partial \mathcal{L}}{\partial(\partial_\tau \phi)}.$$

## **“Arrow of time”:**

Typical motivation:
*"The laws of physics at the fundamental level don’t distinguish between the past and the future."*

**But this is not correct**

The laws of quantum physics have an arrow of causality

Buried in the factors of i in the quantization procedures

Our time convention uses $Z_+$
- if reverse time convention used, use $Z_-$

**Note**: Arrow of thermodynamics follows arrow of causality

## Dueling arrows of causality

Quartic propagators have opposing arrows

$$iD(q^2) \sim \frac{i}{q^2 + i\epsilon} \qquad \text{vs} \quad \sim \frac{-i}{q^2 - M^2 - i\gamma_M}$$

**Who wins?**

-massive state decays

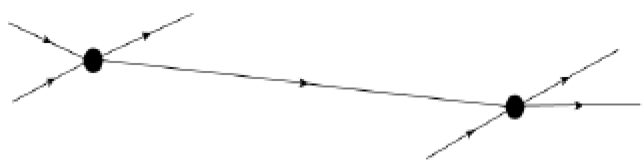
-stable states win

## Living with Causality Uncertainty

Wavepackets are an idealization:
 -really formed by previous interactions

Likewise beam construction from previous scattering
 - and measurement due to final scattering

The timing of scattering will become **uncertain**



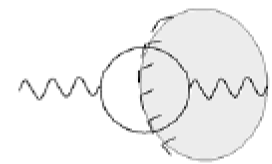
**But causal uncertainty is likely a general property of quantum gravity**

Look for upcoming paper

# Unitarity: Cutkosky cutting rules

Obtain discontinuity by replacing propagator with:

$$\frac{i}{q^2 - m^2 + i\epsilon} \to 2\pi\delta(q^2 - m^2)\theta(q_0)$$



Also on far side of cut, use:

$$\frac{i}{q^2 - m^2 + i\epsilon} \to \frac{-i}{q^2 - m^2 - i\epsilon}$$

Example – self energy

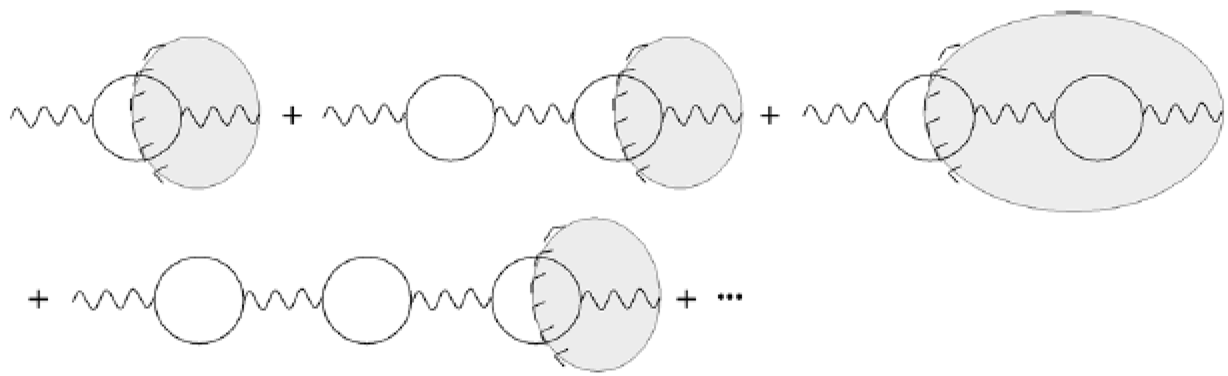
$$\mathrm{Disc}_2\ \Sigma(q) = \frac{\kappa^2 q^4 (N+1)}{2} \int \frac{d^4k}{(2\pi)^4}\ 2\pi\delta(k^2)\theta(k_0)\ 2\pi\delta((q-k)^2)\theta((q-k)_0)$$

Can repackage this:

$$\mathrm{Disc}_2\ \Sigma(q) = 2q\Gamma_2(q)$$

The discontinuity is equivalent to the decay width at $q^2$

# Cuts in a resonance propagator:

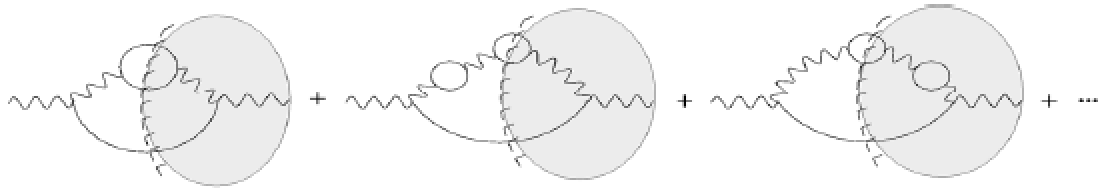


Bubble sum on each side of propagator:
- will c.c. propagators on the far side

$$\text{Disc } D(q) = D(q)\ 2q\Gamma_2(q)\ D^*(q) = -2\ \text{Im}[D(q)]$$

This is true no matter if normal resonance or Merlin modes
- imaginary parts are the same

# Three particle cut = resonance + stable cut



$$\mathrm{Disc}_3\ \Sigma(q) \ = \ \kappa^2 q^4 \int \frac{d^4k_1}{(2\pi)^4}\,\frac{d^4k_2}{(2\pi)^4}\,\frac{1}{(q-k_1)^2 - \frac{\kappa^2(q-k_1)^4}{2\xi^2}}\,\frac{(N+1)\kappa^2(q-k_1)^2}{2}$$
$$\times \quad 2\pi\delta(k_1^2)\theta(k_{10})\ 2\pi\delta(k_2^2)\theta(k_{20})\ 2\pi\delta((q-k_1-k_2)^2)\theta((q-k_1-k_2)_0)\frac{1}{(q-k_1)^2 - \frac{\kappa^2(q-k_1)^4}{2\xi^2}}$$

Identify matrix element

$$\mathcal{M}_3 = \kappa q^2 \kappa (q-k_1)^2 D(q-k_1)$$

and play similar games, to get expected unitarity relation

$$\mathrm{Disc}_3\ \Sigma(q) = 2q\Gamma_3(q)$$

Again result is independent of type of resonance

**Bottom line**: discontinuities come from cuts on stable particles

## Narrow width approximation

Discontinuity in propagator was due to on-shell states only

$$\text{Disc } D(q) = D(q)\ 2q\Gamma(q)\ D^*(q) = \frac{2q\Gamma(q)}{(q^2 - m_r^2)^2 + (m_r\Gamma(q))^2}$$

But when $\Gamma$ is small, this is highly peaked on the resonance,
Use:

$$\delta(x) = \lim_{\epsilon \to 0} \frac{1}{\pi} \frac{\epsilon}{x^2 + \epsilon^2}$$

Limits to usual cutting rule:

$$\lim_{\Gamma \to 0} \text{Disc } D(q) = 2\pi\delta(q^2 - m_r^2)$$

In "three particle cut", this is equiv. decay to resonance plus stable

Again, this result is independent of normal or Merlin resonance

# Lessons ala Veltman

**Physics**:
Cuts for resonances actually are through the stable particles

Resonances do not go on-shell

**Math**:
The $i\gamma$ quickly overwhelms the $i\epsilon$

In the end, this is what Veltman 1963 shows

**Think**: LSZ

$$\langle b|S|a\rangle = \left[i\int d^4x_1 e^{-ip_1\cdot x_1}(\Box_1+m^2)\right]\cdots\left[i\int d^4x_n e^{ip_n\cdot x_n}(\Box_n+m^2)\right]\langle\Omega|T\{\phi(x_1)\cdots\phi(x_n)\}|\Omega\rangle$$

# Heuristic proof of unitarity

Unitarity works with stable particle as external states

Cuts through stable particle loops same for normal and Merlin resonances

Both normal states and Merlin resonances can be in same propagator

Veltman proved normal resonances satisfy unitarity to all orders

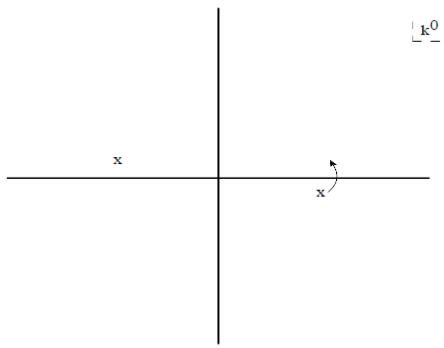
The Merlins will then also satisfy unitarity

# Narrow Width Approximation with Merlin modes

This path follows M. Schwartz: QFT +SM

$$i\mathcal{M} = -\int \frac{d^4k}{(2\pi)^4} \frac{i}{(k-q)^2 + i\epsilon} \frac{-i}{k^2 - m^2 - i\gamma}$$

Normal




Convert $D_F$ to advanced propagator

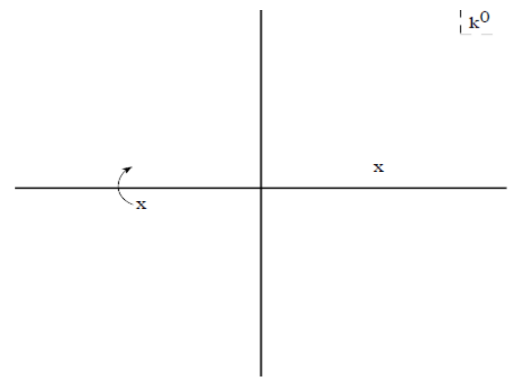
$$\frac{i}{k^2 + i\epsilon} = iD_A(k) + \frac{\pi}{\omega_k}\delta(k_0 - \omega_k)$$

Product of advanced propagators vanishes

Play some games and pick out Im part

$$\mathrm{Im}[\mathcal{M}] = -\int \frac{d^4k}{(2\pi)^4} \left[\pi\delta((k-q)^2)\ \frac{\pi}{E_k}\delta(k_0 - E_k) - \frac{\pi}{\omega_k}\delta(k_0 - q_0 - \omega_k)\ \pi\delta(k^2 - m^2)\right]$$

Ghost



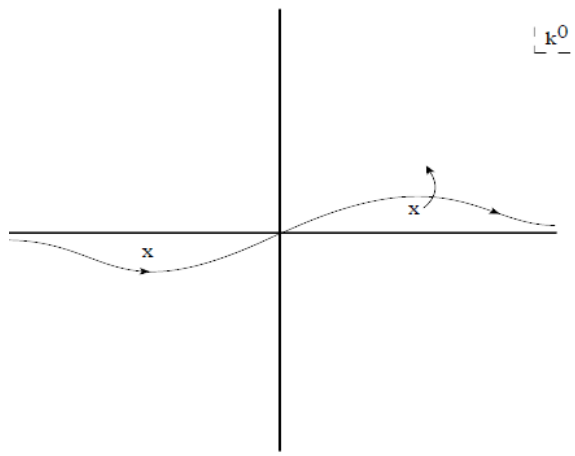

Take same path

$$\frac{-i}{k^2 - m^2 - i\gamma} = -iD_{\text{“}A\text{''}}(k) + \frac{\pi}{E_k}\delta(k_0 + E_{k-p})$$

But delta functions cannot be satisfied

$$\mathrm{Im}[\mathcal{M}] = -\int \frac{d^4k}{(2\pi)^4} \left[\pi\delta((k-q)^2)\ \frac{\pi}{E_k}\delta(k_0 + E_k) - \frac{\pi}{\omega_k}\delta(k_0 - q_0 - \omega_k)\ \pi\delta(k^2 - m^2)\right]$$

# Lee-Wick contour




Contour goes around poles
Returns to normal discontinuity as $\gamma$ goes to zero

Compatible with usual Wick rotation

Treating ghost like a normal particle requires LW contour

## **More needed:**

Stability at higher curvatures/energies
- closer to high mass pole
- if unstable there, is it benign? (like Starobinsky inflation)

More detailed explicit calculations
- Gabriel has one not yet published
- higher order loops

Connection to unitarity-based calculations
-unitarity techniques with unstable particles

Lattice simulations?
- but Euclidean vs. Lorentzian

Etc…

## Summary:

Quadratic gravity is a renormalizeable quantum field theory

**Positive features**:
- massless graviton identified through pole in propagator
- ghost resonance decays – does not appear in spectrum
- seems stable under perturbations
- unitarity with only stable asymptotic states
- LW contour as shortcut via narrow width approximation

**Most unusual feature**:
- causality violation/uncertainty near Planck scale

More work needed, but appears as a viable option for quantum gravity

**Addendum:**

The following slides were not presented in the talk. I wrote them following the discussions at the workshop, in order to be more explicit about some of the issues which concerned some of the other participants. Some of this are comments which I would have presented with extra time, and some is in response to the discussions.

I thank particularly Richard Woodard and Bob Holdom for discussions which sharpened the issues.

This is a very simple toy model which I hope illustrates where the "issues" of quadratic gravity lie, and how Gabriel and I think of quadratic gravity. It was not presented in my talk, but is relevant for the issues which dominated the discussion afterwards. I am tacking it on to the end of slides for the talk in the hopes that it is useful for those folks who revisit the slides after the conference is over.

This starts with the Lagrangian for a higher derivative field $\phi$ coupled to a normal light field $\chi$

$$\mathcal{L}(\phi) = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2M^2}\Box\phi\Box\phi - g\phi\chi^2 \quad . \tag{1}$$

Let us use the path integral

$$Z = \int [d\phi] e^{i\int d^4x \ \mathcal{L}(\phi)} \tag{2}$$

to study the theory.

Let me now manipulate this a bit. I introduce an auxiliary field $\eta$ which when you integrate it out reproduces the same Lagrangian. This is

$$\mathcal{L}(\phi,\eta) = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \eta\Box\phi + \frac{1}{2}M^2\eta^2 - g\phi\chi^2 \quad . \tag{3}$$

As a next step we can define a new field by $\phi = h - \eta$ replacing the field $\phi$ by this combination. The Lagrangian then becomes

$$\begin{aligned}\mathcal{L}(h,\eta) &= \left[\frac{1}{2}\partial_\mu h\partial^\mu h - gh\chi^2\right] \\ &- \left[\frac{1}{2}\partial_\mu\eta\partial^\mu\eta - \frac{1}{2}M^2\eta^2 - g\eta\chi^2\right]\end{aligned} \tag{4}$$

So we have transformed the original theory exactly to

$$\begin{aligned}Z &= \int[dh]e^{i\int d^4x\left[\frac{1}{2}\partial_\mu h\partial^\mu h - gh\chi^2\right]} \\ &\times \int[d\eta]e^{-i\int d^4x\left[\frac{1}{2}\partial_\mu\eta\partial^\mu\eta - \frac{1}{2}M^2\eta^2 - g\eta\chi^2\right]}\end{aligned} \tag{5}$$

The remnant of the original higher derivative term is the $-i$ instead of $+i$ in the second path integral.

What to make of this feature? The second path integral is an acceptable path integral on its own right. It is just the complex conjugate of the usual path integral for a massive particle. Our interpretation of it is as a time-reversed version of the standard path integral, because the Lagrangian itself is T invariant, and T is anti-unitary. This is the only place where $i$ enters the theory when using PI quantization, so the result is just standard QFT with all factors of $i$ going to $-i$. If you were doing canonical quantization you would also want to change the signs of $i$, that is using commutators with $-i\hbar$ instead of $i\hbar$, which is just the Lee-Wick indefinite metric quantization (also similar to Salvio-Strumia. Philip Mannheim's is a bit different.). That form of quantization is designed to also produce positive energy for the free particle states of this particle. But in the end the result in both schemes is that it takes large timelike values of $q^2$ to excite the field at $q^2 = M^2$. It takes twice as much to produce two. It takes yet more to produce two plus a normal particle. This is just like the energy budget of normal particles. This is what is meant by the ghost being a positive energy particle.

In this simple case, it is easy to integrate out the ghost because it is Gaussian. You do this and the result is a factor of

$$e^{\int d^4x d^4y \frac{1}{2} g\chi^2(x)\ iD_{-F}(x-y)\ g\chi^2(y)} \quad . \tag{6}$$

The notation $D_{-F}$ was defined in the talk but is just the complex conjugate of the usual propagator. In the low energy limit, this is just an additive term

$$\Delta\mathcal{L}(\chi) = \frac{g^2}{2M^2}\chi^4 \tag{7}$$

This is a shift in the coupling $\lambda\chi^4$ in the $\chi$ Lagrangian. It is suppressed by $1/M^2$ and so cannot overwhelm the original coupling for large $M$. Further terms coming from the derivative expansion of the propagator are further suppressed by more powers of $1/M^2$. This is Applequist-Carrazonne at work, and is the construction of the low energy effective Lagrangian. The low energy effective Lagrangian just contains $h$ and $\chi$, and is a normal field theory. Since $h$ is massless, it will have a normal classical theory.

This construction has avoided Ostragradsky. The key step that does so is the quantization step (either PI or canonical) that tells us that the ghost is massive with positive energy. That lets us integrate out the massive field. The Ostrogradsky assignment of P's and Q's does not correspond to the quantum construction.

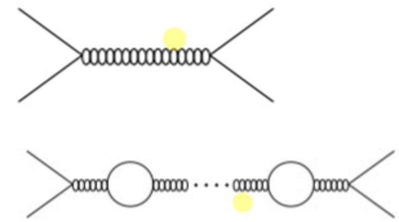
The toy model has features which tell us where to look for problems in both the toy model and in quadratic gravity. The problems are not negative energies, nor Ostrogradsky, nor the classical limit. However, there can still be problems, because you have this heavy particle quantized with the opposite sign of $i$ coupled to normal particles with the usual sign of $i$. This gets excited at high energy by the scattering of two $\chi$ field. That these carry positive energy to produce this resonance confirms the positive energy interpretation of $\eta$. However there is also the minus sign in the propagator near the massive pole. That is where the problem lies, not in negative energies nor in the classical limit. It becomes the "dueling arrows of causality" issue which Gabriel and I focussed on. At this stage the reader should go back and follow the development from the section titled "Spectrum". We there discuss the cases with high energy quantum behavior. The main result is the violation of micro-causality which comes from the differing factors of $i$.

I am also attaching to the talk three slides which I cut at the last minute because I felt (correctly) that I would not have time to present them. These are an explicit example of how unitarity is satisfied when scattering through the ghost-like pole. The factors of $i$ work out just right to make the amplitude unitary. These slides were originally in the talk immediately following the slide "Heuristic proof of unitarity", and are in the same notation as the earlier slides on the spectrum of the theory.

If you have comments about this addendum, I would be happy to hear them.

# Unitarity in the spin two channel

Do these features cause trouble in scattering?
- consider scattering in spin 2 channel



First consider single scalar at low energy:

$$i\mathcal{M} = \left(\frac{1}{2}V_{\mu\nu}(q)\right)\left[iD^{\mu\nu\alpha\beta}(q^2)\right]\left(\frac{1}{2}V_{\alpha\beta}(-q)\right)$$

$$\mathcal{M} = 16\pi\sum_{J=0}^{\infty}(2J+1)T_J(s)P_J(\cos\theta)$$

Results in

$$T_2(s) = -\frac{N_{\rm eff}s}{640\pi}\bar{D}(s).$$

$N_{\rm eff} = 1/6$ for a single scalar field

$$\bar{D}^{-1}(s) = \frac{1}{\tilde{\kappa}^2}\left\{1 - \frac{\tilde{\kappa}^2 s}{2\xi^2(\mu)} - \frac{\tilde{\kappa}^2 s N_{\rm eff}}{640\pi^2}\ln\left(\frac{s}{\mu^2}\right) + \frac{i\tilde{\kappa}^2 s N_{\rm eff}}{640\pi}\right\}$$

## Satisfies elastic unitarity:

$$\mathrm{Im} T_2 = |T_2|^2.$$

This implies the structure

$$T_2(s) = \frac{A(s)}{f(s) - iA(s)} = \frac{A(s)[f(s) + iA(s)]}{f^2(s) + A^2(s)}$$

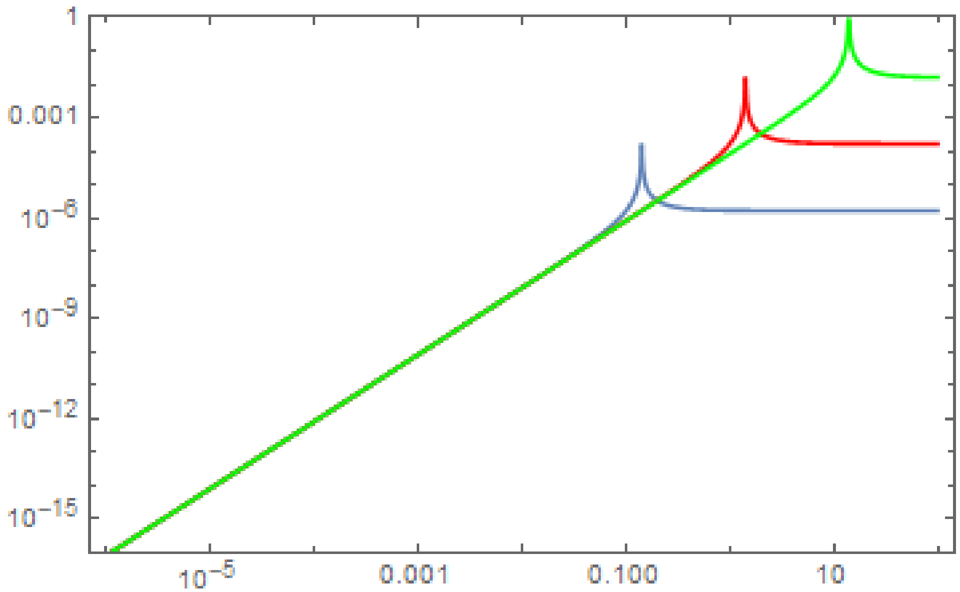
for any real *f(s)*

Signs and magnitudes work out for $A(s) = -\frac{N_{\mathrm{eff}} s}{640\pi}.$

**Multi-particle problem:**

- just diagonalize the J=2 channel
- same result but with general N

# Scattering amplitude at weak coupling:




$\xi^2 = 0.1, 1, 10$